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On the smooth characters of the additive group of a local p-field

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Abstract.We build all smooth characters of a local p-field.

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1. Introduction and Notation

In p-adic Representation Theory a non-trivial Character of the additive group of the ground field is needed. Therefor, the existence of a non-trivial character with a given conductor of the additive group of the ground field is important. Here we build all smooth characters of a local p-field.

Let p be a prime number and let F be a local p-field. Thus F is a locally compact, totally disconnected topological field and its ring of integers, O, is a compact open set. Let P be the maximal ideal of O and let π be its a generator. For an integer n set

$$P^n = O\pi^n = \{x\pi^n \,|x \in O\}$$

Then $\{P^n \,|n \in \mathbb{Z}\}$ is a fundamental system of neighborhoods of zero in F. Let $v(x)$ denotes the order of x in F. Thus for any $x \in F$ we have $x = \varepsilon\pi^{v(x)}$ where ε is a unit element in O. Let F^+ be the additive group of F. It is obvious that we have

$$F^+ \supset \cdots P^{n-1} \supset \cdots \supset P^{-1} \supset O \supset P \supset \cdots \supset \{0\}$$

and

$$F = \bigcup_{n\in\mathbb{Z}} P^n$$

Let $\mathbb{C}^\times$ denote the set of non-zero complex numbers.

Definition 1. *A function $\chi : F^+ \to \mathbb{C}^\times$ is called a character of F^+ if for all $x, y \in F^+$ the following holds*

$$\chi(x+y) = \chi(x)\,\chi(y)$$

The conductor of χ *is the smallest integer,* n *for which* χ *is trivial on* P^n*, i.e.* $\chi(x) = 1$*, for all* $x \in P^n$*, but* $\chi(x) \neq 1$ *at least for one* $x \in P^{n-1}$*.*

2. Existence of a non-triviall character

We will construct a non-trivial character for F^+ with the conductor 0, i.e. a character which is trivial on O and non-trivial everywhere else. Since P^n is compact for each $n \in \mathbb{Z}$, and O is open, P^{-n}/O is finite for any $n > 0$. Since $\mathbb{C}^\times$ contains all roots of unity we can define a character $\xi_n : P^{-n}/O \to \mathbb{C}^\times$.

Proposition 1. *Let* $\chi_n : P^{-n} \to \mathbb{C}^\times$ *be a map defined by* $\chi_n(x) = \xi_n(x + O)$*. Then* χ_n *is a character of* P^{-n} *and is trivial on* O*.*

Proof. This is obvious because ξ_n is a character and trivial on O. □

Now let χ_{n+1} be an extension of χ_n to P^{-n-1}.(This is possible because P^{-n-1}, P^{-n} and $\mathbb{C}^\times$ are all $\mathbb{Z}$-modules and $\mathbb{C}^\times$ is injective.)

Proposition 2. *Define* $\chi : F^+ \to \mathbb{C}^\times$ *as follows: Let* $x \in F$*. Then there is an integer* $n > 0$ *such that* $x \in P^{-n}$*. Now set* $\chi(x) = \chi_n(x)$*. Then* χ *is a non-trivial character of* F^+ *and its conductor is* 0*.*

Proof. Given that $x, y \in F$, then there is an integer $n > 0$ such that $x, y \in P^{-n}$. Thus $x + y \in P^{-n}$ and we have:

$$\begin{aligned} \chi(x+y) &= \chi_n(x+y) \\ &= \chi_n(x)\chi_n(y) \\ &= \chi(x)\chi(y). \end{aligned}$$

Here we used this fact that χ_n is an extension of χ_{n-1} for each $n > 0$. Since each χ_n is trivial on O and non-trivial everywhere else χ has 0 conductor. □

Let χ be a nontrivial character of F^+. For any $a \in F$, define $\chi_a : F^+ \to \mathbb{C}^\times$ by $\chi_a(x) = \chi(ax)$. One can show that χ_a is a character of F^+.

Lemma 1. *Let* χ *be a nontrivial character of* F^+ *whose conductor is* 0*. Let* ψ *be a character of* F^+ *whose conductor is* $k > 0$*. For a given integer* $n > 0$*, there is an element* $a_n \in P^{-k}$ *such that* $\psi = \chi_{a_n}$ *on* P^{-n}*, i.e.* $\psi(x) = \chi(a_n x)$ *for all* $x \in P^{-n}$*.*

Proof. Since P^{-n} is compact and $\ker\chi$ and $\ker\psi$ are open, so $P^{-n}/(P^{-n} \cap \ker\chi)$ and $P^{-n}/(P^{-n} \cap \ker\psi)$ are finite. Thus $\chi(P^{-n})$ and $\psi(P^{-n})$ are finite subgroups of $\mathbb{C}^\times$ and hence they are cyclic. Let $m_1 = |\chi(P^{-n})|$ and $m_2 = |\psi(P^{-n})|$, and let m be the least common multiple of m_1 and

m_2. Let $\omega \in \mathbb{C}^\times$ be a complex root of $\omega^m = 1$. Now set $\alpha = \frac{m}{m_1}$ and $\beta = \frac{m}{m_2}$. Then we have $\chi(P^{-n}) = \langle \omega^\alpha \rangle$ and $\psi(P^{-n}) = \langle \omega^\beta \rangle$. From here we get $\chi(\pi^{-n}) = \omega^\alpha$, and $\psi(\pi^{-n}) = \omega^\beta$. Thus

$$\begin{aligned} \chi\left(m_1\pi^{-n}\right) &= \omega^{m_1\alpha} \\ &= \omega^m \\ &= 1 \\ &= \omega^m \\ &= \omega^{m_2\beta} \\ &= \psi\left(m_2\pi^{-n}\right). \end{aligned}$$

Now from $\chi(m_1\pi^{-n}) = \psi(m_2\pi^{-n})$, one can get $\psi(\pi^{-n}) = \chi\left(\frac{m_1}{m_2}\pi^{-n}\right)$ and consequent $\psi(x) = \chi\left(\frac{m_1}{m_2}x\right)$, for all $x \in P^{-n}$. Now set $a_n = \frac{m_1}{m_2}$. Since $P^k \subset P^{-n}$, for $x \in P^k$ we have

$$\begin{aligned} 1 &= \psi(x) \\ &= \chi(a_n x) \end{aligned}$$

Thus $a_n x \in O$. From here we get $a_n \in P^{-k}$. □

Theorem 1. *Let χ be a nontrivial character of F^+ whose conductor is 0. For any character ψ of F^+ with the conductor $k > 0$ there is a unique $a \in F$ such that $\psi = \chi_a$.*

Proof. By Lemma 1 we have $\psi(x) = \chi(a_n x)$ on P^{-n} for some $a_n \in P^{-k}$. Thus by the same Lemma we have $\psi(x) = \chi(a_{n+1}x)$ on P^{-n-1} for some $a_{n+1} \in P^{-k}$. From here for all $x \in P^{-n}$ we have

$$\begin{aligned} \chi(a_n x) &= \psi(x) \\ &= \chi(a_{n+1}x). \end{aligned}$$

Thus $\chi(a_n x) = \chi(a_{n+1}x)$ or $\chi((a_{n+1} - a_n)x) = 1$. This last equation implies that $(a_{n+1} - a_n)x \in O$ for all $x \in P^{-n}$ and all $n > 0$. Since the order of a_n does not depend on n we must have $a_{n+1} - a_n = 0$ or $a_{n+1} = a_n = a$. Thus $\psi = \chi_a$. To show a is unique it is enough to show if $\chi_a(x) = 1$ for all $x \in F$, then $a = 0$. Finally, suppose $\chi_a(x) = \chi(ax) = 1$ for all $x \in F$ but $a \neq 0$. For this a, one can show that $aF = F$; implying that χ must be trivial, which it is not. □

References

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